

INEQUALITIES FOR TRACE ON τ -MEASURABLE OPERATORS

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ABSTRACT. Let \mathfrak{M} be a semifinite von Neumann algebra on a Hilbert space equipped with a faithful normal semifinite trace τ . A closed densely defined operator x affiliated with \mathfrak{M} is called τ -measurable if there exists a number $\lambda \geq 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$. A number of useful inequalities, which are known for the trace on Hilbert space operators, are extended to trace on τ -measurable operators. In particular, these inequalities imply Clarkson inequalities for n -tuples of τ -measurable operators. A general parallelogram law for τ -measurable operators are given as well.

1. INTRODUCTION AND PRELIMINARIES

Let \mathfrak{M} be a semifinite von Neumann algebra on a Hilbert space \mathfrak{H} , with unit element $\mathbf{1}$, equipped with a faithful normal semifinite trace τ . For standard facts concerning von Neumann algebras, we refer the reader to [2]. A closed densely defined linear operator $x : \mathcal{D}(x) \subseteq \mathfrak{H} \rightarrow \mathfrak{H}$ is called affiliated with \mathfrak{M} if $ux = xu$ for all unitaries u in the commutant \mathfrak{M}' of \mathfrak{M} . If x is in the algebra $\mathcal{B}(\mathfrak{H})$ of all bounded linear operators on the Hilbert space \mathfrak{H} , then x is affiliated with \mathfrak{M} if and only if $x \in \mathfrak{M}$. A closed densely defined operator x affiliated with \mathfrak{M} is called τ -measurable if there exists a number $\lambda \geq 0$ such that

$$\tau(e^{|x|}(\lambda, \infty)) < \infty,$$

where $|x| = (x^*x)^{1/2}$ and e^a denotes the spectral measure of the self-adjoint operator a which is a σ -additive (w.r.t. the strong operator topology) from the Borel σ -algebra of \mathbb{R} into the orthogonal projections. The collection of all τ -measurable operators is denoted by $\widetilde{\mathfrak{M}}$. With the sum and product defined as the respective closure of the algebraic sum and product, it is well known that $\widetilde{\mathfrak{M}}$ is a $*$ -algebra with respect to the operations of strong sum and strong product and taking adjoint; see [11]. For $x \in \widetilde{\mathfrak{M}}$, the generalized singular value function

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$\mu(x) : [0, \infty] \rightarrow [0, \infty]$ is defined by

$$\mu_t(x) = \inf \{ \lambda \geq 0 : \tau(e^{|x|}(\lambda, \infty)) \leq t \}, \quad t \geq 0.$$

The generalized singular value function $\mu(x)$ is decreasing right-continuous. Moreover,

$$\mu(uxv) \leq \|u\| \|v\| \mu(x)$$

for all $u, v \in \mathfrak{M}$ and $x \in \widetilde{\mathfrak{M}}$. Moreover,

$$\mu(f(x)) = f(\mu(x))$$

whenever $0 \leq x \in \widetilde{\mathfrak{M}}$ and f is an increasing continuous function on $[0, \infty)$ satisfying $f(0) = 0$. The space $\widetilde{\mathfrak{M}}$ is a partially ordered vector space under the ordering $x \geq 0$ defined by $\langle x\xi, \xi \rangle \geq 0$, $\xi \in \mathcal{D}(x)$. The trace τ on \mathfrak{M}^+ extends uniquely to an additive, positively homogeneous, unitarily invariant and normal functional $\tilde{\tau} : \widetilde{\mathfrak{M}} \rightarrow [0, \infty]$, which is given by $\tilde{\tau}(x) = \int_0^\infty \mu_t(x) dt$, $x \in \mathfrak{M}^+$ [4]. This extension is also denoted by τ . Further,

$$\tau(f(x)) = \int_0^\infty f(\mu_t(x)) dt \tag{1.1}$$

whenever $0 \leq x \in \widetilde{\mathfrak{M}}$ and f is non-negative Borel function which is bounded on a neighborhood of 0 and satisfies $f(0) = 0$. For $0 < p < \infty$, $L^p(\mathfrak{M}, \tau)$ is defined as the set of all densely defined closed operators x affiliated with \mathfrak{M} such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} = \left(\int_0^\infty \mu_t(x)^p dt \right)^{\frac{1}{p}} < \infty.$$

For further details and proofs, we refer the reader to [15, 5, 3, 12, 13]. Important special cases of these noncommutative spaces are usual L^p -spaces and the Schatten p -classes \mathcal{C}_p .

The classical Clarkson inequalities, for Schatten p -norms of Hilbert space operators, assert that if $A, B \in \mathcal{B}(\mathfrak{H})$, then

$$2(\|A\|_p^p + \|B\|_p^p) \leq \|A + B\|_p^p + \|A - B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p) \tag{1.2}$$

for $2 \leq p < \infty$, and

$$2^{p-1}(\|A\|_p^p + \|B\|_p^p) \leq \|A + B\|_p^p + \|A - B\|_p^p \leq 2(\|A\|_p^p + \|B\|_p^p) \tag{1.3}$$

for $0 < p \leq 2$ (see [6]). Bahatia and Kittaneh [1] generalized the Clarkson inequalities (1.2) and (1.3) for n -tuples operators as follows. If $A_0, \dots, A_{n-1} \in \mathcal{B}(\mathfrak{H})$ and $\omega_0, \dots, \omega_{n-1}$ are n roots of unity with $\omega_j = e^{2\pi i j/n}$, $0 \leq j \leq n-1$, then

$$n \sum_{j=0}^{n-1} \|A_j\|_p^p \leq \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k A_j \right\|_p^p \leq n^{p-1} \sum_{j=0}^{n-1} \|A_j\|_p^p \quad (1.4)$$

for $2 \leq p \leq \infty$, and

$$n^{p-1} \sum_{j=0}^{n-1} \|A_j\|_p^p \leq \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k A_j \right\|_p^p \leq n \sum_{j=0}^{n-1} \|A_j\|_p^p \quad (1.5)$$

for $0 < p < \infty$. Related Clarkson inequalities for n -tuples of operators have been recently given by Kissin [8] and Hirzalleh and Kitaneh [7]. These inequalities have been found to be very powerful tools in operator theory and in mathematical physics (see, e.g., [14]). In [5], Fack and Kosaki proved Clarkson inequalities for measurable operators in Haagerup L^p -spaces. Let x be a τ -measurable operator in $L^p(\mathfrak{M}, \tau)$. Since $\|x\|_p^p = \tau(|x|^p)$, for $0 < p < \infty$, our generalization of the inequalities (1.2) and (1.3) will be clear if we rewrite them as

$$2[\tau(|x|^p) + \tau(|y|^p)] \leq \tau(|x+y|^p) + \tau(|x-y|^p) \leq 2^{p-1}[\tau(|x|^p) + \tau(|y|^p)] \quad (1.6)$$

for $2 \leq p < \infty$, and

$$2^{p-1}[\tau(|x|^p) + \tau(|y|^p)] \leq \tau(|x+y|^p) + \tau(|x-y|^p) \leq 2[\tau(|x|^p) + \tau(|y|^p)] \quad (1.7)$$

for $0 < p \leq 2$.

There are several extensions of the classical parallelogram law in the literature. Generalizations of the parallelogram law for the Schatten p -norms have been given in the form of the celebrated Clarkson inequalities; see [7, 9, 10] and references cited therein.

A number of useful inequalities relating the traces of operators on a Hilbert space are known when the trace is defined in the usual way. In this paper, we present some inequalities for trace on τ -measurable operators. These inequalities to trace generalized Clarkson inequalities for n -tuples of τ -measurable operators. In fact, we give natural generalizations of inequalities (1.6) and (1.7) for n -tuples of τ -measurable operators and show that the power functions $\varphi(t) = t^p$ can be replaced by more general classes of functions. A general parallelogram law for τ -measurable operators are given as well.

2. TRACE INEQUALITIES FOR n -TUPLE OF τ -MEASURABLE OPERATORS

In this section, we give new trace inequalities for n -tuple of τ -measurable operators that considerably generalize inequalities (1.6) and (1.7). Before we give the main theorems of this section, we need the following lemma, which clearly is an extension of [5, proposition 4.6]) for n -tuples of τ -measurable operators.

Lemma 2.1. *Let x_0, \dots, x_{n-1} be positive τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j = 1$. Then, for every continuous increasing function f on \mathbb{R}_+ with $f(0) = 0$,*

$$\tau \left(f \left(\sum_{j=0}^{n-1} \alpha_j x_j \right) \right) \leq \sum_{j=0}^{n-1} \alpha_j \tau(f(x_j)) \quad \text{when } f \text{ is convex.} \quad (2.1)$$

$$\sum_{j=0}^{n-1} \alpha_j \tau(f(x_j)) \leq \tau \left(f \left(\sum_{j=0}^{n-1} \alpha_j x_j \right) \right) \quad \text{when } f \text{ is concave.} \quad (2.2)$$

$$\sum_{j=0}^{n-1} \tau(f(x_j)) \leq \tau \left(f \left(\sum_{j=0}^{n-1} x_j \right) \right) \quad \text{when } f \text{ is convex.} \quad (2.3)$$

$$\tau \left(f \left(\sum_{j=0}^{n-1} x_j \right) \right) \leq \sum_{j=0}^{n-1} \tau(f(x_j)) \quad \text{when } f \text{ is concave.} \quad (2.4)$$

Now we are in a position to present our main result, providing a generalization of inequality (1.6) for n -tuples of τ -measurable operators.

Theorem 2.2. *Let x_0, \dots, x_{n-1} be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j = 1$. Let φ be a continuous increasing function on \mathbb{R}_+ with $\varphi(0) = 0$ such that $\psi(t) = \varphi(\sqrt{t})$ is convex. Then*

$$\tau \left(\varphi \left(\left| \sum_{j=0}^{n-1} \alpha_j x_j \right| \right) \right) + \sum_{0 \leq j < k \leq n-1} \tau \left(\varphi \left(\sqrt{\alpha_j \alpha_k} |x_j - x_k| \right) \right) \leq \sum_{j=0}^{n-1} \alpha_j \tau(\varphi(|x_j|)). \quad (2.5)$$

Proof. The following useful identity can be easily verified,

$$\left| \sum_{j=0}^{n-1} \alpha_j x_j \right|^2 + \sum_{0 \leq j < k \leq n-1} \alpha_j \alpha_k |x_j - x_k|^2 = \sum_{j=0}^{n-1} \alpha_j |x_j|^2. \quad (2.6)$$

Now, by using identity (2.6) and inequalities (2.1), (2.3) we get

$$\begin{aligned}
\sum_{j=0}^{n-1} \alpha_j \tau(\varphi(|x_j|)) &= \sum_{j=0}^{n-1} \alpha_j \tau(\psi(|x_j|^2)) \\
&\geq \tau \left(\psi \left(\sum_{j=0}^{n-1} \alpha_j |x_j|^2 \right) \right) \\
&= \tau \left(\psi \left(\left| \sum_{j=0}^{n-1} \alpha_j x_j \right|^2 + \sum_{0 \leq j < k \leq n-1} \alpha_j \alpha_k |x_j - x_k|^2 \right) \right) \\
&\geq \tau \left(\psi \left(\left| \sum_{j=0}^{n-1} \alpha_j x_j \right|^2 \right) \right) + \sum_{0 \leq j < k \leq n-1} \tau(\psi(\alpha_j \alpha_k |x_j - x_k|^2)) \\
&= \tau \left(\varphi \left(\left| \sum_{j=0}^{n-1} \alpha_j x_j \right|^2 \right) \right) + \sum_{0 \leq j < k \leq n-1} \tau(\varphi(\sqrt{\alpha_j \alpha_k} |x_j - x_k|)).
\end{aligned}$$

□

Based on Lemma 2.1 and inequalities (2.2) and (2.4), one can employ an argument similar to that used in the proof of Theorem 2.2 to prove the following related result.

Theorem 2.3. *Let x_0, \dots, x_{n-1} be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j = 1$. Let φ be a continuous increasing function on \mathbb{R}_+ with $\varphi(0) = 0$ such that $\psi(t) = \varphi(\sqrt{t})$ is concave. Then*

$$\sum_{j=0}^{n-1} \alpha_j \tau(\varphi(|x_j|)) \leq \tau \left(\varphi \left(\left| \sum_{j=0}^{n-1} \alpha_j x_j \right| \right) \right) + \sum_{0 \leq j < k \leq n-1} \tau(\varphi(\sqrt{\alpha_j \alpha_k} |x_j - x_k|)). \quad (2.7)$$

Applying Theorem 2.2 and Theorem 2.3 for the functions $\varphi(t) = t^p$ ($2 \leq p < \infty$) and $\varphi(t) = t^p$ ($0 < p \leq 2$), respectively, we obtain natural generalization of Clarkson inequalities for n -tuples of τ -measurable operators.

Corollary 2.4. *Let x_0, \dots, x_{n-1} be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j = 1$. Then*

$$\left\| \sum_{j=0}^{n-1} \alpha_j x_j \right\|_p^p + \sum_{0 \leq j < k \leq n-1} (\alpha_j \alpha_k)^{\frac{p}{2}} \|x_j - x_k\|_p^p \leq \sum_{j=0}^{n-1} \alpha_j \|x_j\|_p^p$$

for $2 \leq p < \infty$, and

$$\sum_{j=0}^{n-1} \alpha_j \|x_j\|_p^p \leq \left\| \sum_{j=0}^{n-1} \alpha_j x_j \right\|_p^p + \sum_{0 \leq j < k \leq n-1} (\alpha_j \alpha_k)^{\frac{p}{2}} \|x_j - x_k\|_p^p$$

for $0 < p \leq 2$. In particular

$$\left\| \sum_{j=0}^{n-1} x_j \right\|_p^p + \sum_{0 \leq j < k \leq n-1} \|x_j - x_k\|_p^p \leq n^{p-1} \sum_{j=0}^{n-1} \|x_j\|_p^p$$

for $2 \leq p < \infty$, and

$$n^{p-1} \sum_{j=0}^{n-1} \|x_j\|_p^p \leq \left\| \sum_{j=0}^{n-1} x_j \right\|_p^p + \sum_{0 \leq j < k \leq n-1} \|x_j - x_k\|_p^p$$

for $0 < p \leq 2$.

Let $\varphi(t) = e^{t^2} - 1$. Then φ is a continuous increasing function on \mathbb{R}_+ , $\varphi(0) = 0$ and $\psi(t) = \varphi(\sqrt{t}) = e^t - 1$ is convex. Applying Theorem 2.2 to this spacial function, we get the next result.

Corollary 2.5. *Let x_0, \dots, x_{n-1} be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j = 1$. Then*

$$\tau \left(e^{|\sum_{j=0}^{n-1} \alpha_j x_j|^2} - \mathbf{1} \right) + \sum_{0 \leq j < k \leq n-1} \tau \left(e^{\alpha_j \alpha_k |x_j - x_k|^2} - \mathbf{1} \right) \leq \sum_{j=0}^{n-1} \alpha_j \tau \left(e^{|x_j|^2} - \mathbf{1} \right).$$

Now let $\varphi(t) = \log(t+1)$. Then $\psi(t) = \varphi(\sqrt{t}) = \log(\sqrt{t}+1)$ is concave. Applying Theorem 2.3 to this function, we obtain the following corollary.

Corollary 2.6. *Let x_0, \dots, x_{n-1} be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j = 1$. Then*

$$\sum_{j=0}^{n-1} \alpha_j \tau (\log(|x_j| + 1)) \leq \tau \left(\log \left(\left| \sum_{j=0}^{n-1} \alpha_j x_j \right| + 1 \right) \right) + \sum_{0 \leq j < k \leq n-1} \tau \left(\log \left(\sqrt{\alpha_j \alpha_k} |x_j - x_k| + 1 \right) \right).$$

3. REFINEMENT OF TRACE INEQUALITIES

In this section, by using some trace inequalities for n -tuples of τ -measurable operators, we obtain a refinement of the Clarkson inequalities, which was established by Bahatia and Kittaneh in [1]. To this end, we need the following useful identity

$$\frac{1}{n} \sum_{k=0}^{n-1} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right|^2 = \sum_{j=0}^{n-1} |x_j|^2, \quad (3.1)$$

where $\omega_0, \dots, \omega_{n-1}$ are the n roots of unity with $\omega_j = e^{2\pi i j/n}$, $0 \leq j \leq n-1$ (see [1]).

Theorem 3.1. *Let x_0, \dots, x_{n-1} be τ -measurable operators and let φ be a continuous increasing function on \mathbb{R}_+ with $\varphi(0) = 0$ such that $\psi(t) = \varphi(\sqrt{t})$ is convex. Then*

$$\begin{aligned} \sum_{k=0}^{n-1} \tau \left(\varphi \left(\frac{1}{\sqrt{n}} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right| \right) \right) &\leq \tau \left(\varphi \left(\left(\sum_{j=0}^{n-1} |x_j|^2 \right)^{\frac{1}{2}} \right) \right) \\ &\leq \frac{1}{n} \sum_{k=0}^{n-1} \tau \left(\varphi \left(\left| \sum_{j=0}^{n-1} \omega_j^k x_j \right| \right) \right). \end{aligned} \quad (3.2)$$

Proof. Using inequalities (2.1), (2.3) and identity (3.1), we reach the first inequality in (3.2).

$$\begin{aligned} \sum_{k=0}^{n-1} \tau \left(\varphi \left(\frac{1}{\sqrt{n}} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right| \right) \right) &= \sum_{k=0}^{n-1} \tau \left(\psi \left(\frac{1}{n} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right|^2 \right) \right) \\ &\leq \tau \left(\psi \left(\frac{1}{n} \sum_{k=0}^{n-1} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right|^2 \right) \right) \\ &= \tau \left(\psi \left(\sum_{j=0}^{n-1} |x_j|^2 \right) \right) \\ &= \tau \left(\varphi \left(\left(\sum_{j=0}^{n-1} |x_j|^2 \right)^{\frac{1}{2}} \right) \right), \end{aligned}$$

as desired.

Now using Lemma 2.1, inequality (2.1) and identity (3.1) we get the second inequality in (3.2).

$$\begin{aligned}
\tau \left(\varphi \left(\left(\sum_{j=0}^{n-1} |x_j|^2 \right)^{\frac{1}{2}} \right) \right) &= \tau \left(\varphi \left(\left(\frac{1}{n} \sum_{k=0}^{n-1} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right|^2 \right)^{\frac{1}{2}} \right) \right) \\
&= \tau \left(\psi \left(\frac{1}{n} \sum_{k=0}^{n-1} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right|^2 \right) \right) \\
&\leq \frac{1}{n} \sum_{k=0}^{n-1} \tau \left(\psi \left(\left| \sum_{j=0}^{n-1} \omega_j^k x_j \right|^2 \right) \right) \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \tau \left(\varphi \left(\left| \sum_{j=0}^{n-1} \omega_j^k x_j \right| \right) \right).
\end{aligned}$$

□

Based on inequalities (2.2), (2.4) and identity (3.1), one can employ an argument similar to that used in the proof of Theorem 3.1 to get the following related result.

Theorem 3.2. *Let x_0, \dots, x_{n-1} be τ -measurable operators and let φ be a continuous increasing function on \mathbb{R}_+ with $\varphi(0) = 0$ such that $\psi(t) = \varphi(\sqrt{t})$ is concave. Then*

$$\frac{1}{n} \sum_{k=0}^{n-1} \tau \left(\varphi \left(\left| \sum_{j=0}^{n-1} \omega_j^k x_j \right| \right) \right) \leq \tau \left(\varphi \left(\left(\sum_{j=0}^{n-1} |x_j|^2 \right)^{\frac{1}{2}} \right) \right) \leq \sum_{k=0}^{n-1} \tau \left(\varphi \left(\frac{1}{\sqrt{n}} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right| \right) \right).$$

Applying Theorem 3.1 and Theorem 3.2 for the functions $\varphi(t) = t^p$ ($2 \leq p < \infty$) and $\varphi(t) = t^p$ ($0 < p \leq 2$), respectively, we obtain refinements of the Clarkson inequalities for n -tuples of τ -measurable operators.

Corollary 3.3. *Let x_0, \dots, x_{n-1} be τ -measurable operators. Then*

$$n^{-\frac{p}{2}} \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k x_j \right\|_p^p \leq \left\| \left(\sum_{j=0}^{n-1} |x_j|^2 \right)^{\frac{1}{2}} \right\|_p^p \leq \frac{1}{n} \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k x_j \right\|_p^p$$

for $2 \leq p < \infty$, and

$$\frac{1}{n} \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k x_j \right\|_p^p \leq \left\| \left(\sum_{j=0}^{n-1} |x_j|^2 \right)^{\frac{1}{2}} \right\|_p^p \leq n^{-\frac{p}{2}} \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k x_j \right\|_p^p$$

for $0 < p \leq 2$.

Applying Theorem 3.1 and Theorem 3.2 to special functions $\varphi(t) = e^{t^2} - 1$ and $\varphi(t) = \log(t + 1)$ respectively, we obtain the following results.

Corollary 3.4. *Let x_0, \dots, x_{n-1} be τ -measurable operators. Then*

$$\sum_{k=0}^{n-1} \tau \left(e^{\frac{1}{n} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right|^2} - \mathbf{1} \right) \leq \tau \left(e^{\sum_{j=0}^{n-1} |x_j|^2} - \mathbf{1} \right) \leq \frac{1}{n} \sum_{k=0}^{n-1} \tau \left(e^{\left| \sum_{j=0}^{n-1} \omega_j^k x_j \right|^2} - \mathbf{1} \right).$$

Corollary 3.5. *Let x_0, \dots, x_{n-1} be τ -measurable operators. Then*

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} \tau \left(\log \left(\left| \sum_{j=0}^{n-1} \omega_j^k x_j \right| + \mathbf{1} \right) \right) &\leq \tau \left(\log \left(\left(\sum_{j=0}^{n-1} |x_j| \right)^{\frac{1}{2}} + \mathbf{1} \right) \right) \\ &\leq \frac{1}{n} \sum_{k=0}^{n-1} \tau \left(\log \left(\frac{1}{n} \left| \sum_{j=0}^{n-1} \omega_j^k x_j \right| + \mathbf{1} \right) \right). \end{aligned}$$

4. A GENERAL PARALLELOGRAM LAW FOR τ -MEASURABLE OPERATORS

The parallelogram law for n -tuples of operators has been recently investigated by the second author [9]. In this section, by using some trace inequalities for n -tuples of τ -measurable operators, we obtain an extension of the main result of [9] for n -tuples of τ -measurable operators. Let $x_0, \dots, x_{n-1}, y_1, \dots, y_n$ be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers. Then the following identity can be given similar to [9, Corollary 2.3],

$$\begin{aligned} \sum_{0 \leq i < j \leq n-1} \left| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right|^2 + \sum_{0 \leq i < j \leq n-1} \left| \sqrt{\frac{\alpha_i}{\alpha_j}} y_i - \sqrt{\frac{\alpha_j}{\alpha_i}} y_j \right|^2 \\ = \sum_{i,j=0}^{n-1} \left| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} y_j \right|^2 - \left| \sum_{i=0}^{n-1} (x_i - y_i) \right|^2. \end{aligned} \quad (4.1)$$

If we set $y_1 = \dots = y_n = 0$ in (4.1) and $\sum_{j=0}^{n-1} \frac{1}{\alpha_j} = 1$, then the following identity can be given

$$\sum_{0 \leq i < j \leq n-1} \left| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right|^2 = \sum_{i=0}^{n-1} \alpha_i |x_i|^2 - \left| \sum_{i=0}^{n-1} x_i \right|^2. \quad (4.2)$$

Theorem 4.1. *Let $x_0, \dots, x_{n-1}, y_1, \dots, y_n$ be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real number such that $\sum_{i,j=0}^{n-1} \frac{1}{\sqrt{\alpha_i \alpha_j}} = 1$. Let φ be a continuous increasing function*

on \mathbb{R}_+ with $\varphi(0) = 0$ such that $\psi(t) = \varphi(\sqrt{t})$ is convex. Then

$$\begin{aligned} \sum_{i,j=0}^{n-1} \frac{1}{\sqrt{\alpha_i \alpha_j}} \tau(\varphi(|\alpha_i x_i - \alpha_j y_j|)) &\geq \\ \sum_{0 \leq i < j \leq n-1} \tau \left(\varphi \left(\left| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right| \right) \right) &+ \sum_{0 \leq i < j \leq n-1} \tau \left(\varphi \left(\left| \sqrt{\frac{\alpha_i}{\alpha_j}} y_i - \sqrt{\frac{\alpha_j}{\alpha_i}} y_j \right| \right) \right) \\ &+ \tau \left(\varphi \left(\left| \sum_{i=0}^{n-1} (x_i - y_i) \right| \right) \right). \end{aligned} \quad (4.3)$$

If $\psi(t)$ is concave, then the reverse of inequality (4.3) holds.

Proof.

$$\begin{aligned} &\sum_{i,j=0}^{n-1} \frac{1}{\sqrt{\alpha_i \alpha_j}} \tau(\varphi(|\alpha_i x_i - \alpha_j y_j|)) \\ &= \sum_{i,j=0}^{n-1} \frac{1}{\sqrt{\alpha_i \alpha_j}} \tau(\psi(|\alpha_i x_i - \alpha_j y_j|^2)) \\ &\geq \tau \left(\psi \left(\sum_{i,j=0}^{n-1} \frac{1}{\sqrt{\alpha_i \alpha_j}} |\alpha_i x_i - \alpha_j y_j|^2 \right) \right) \quad (\text{by (2.1)}) \\ &= \tau \left(\psi \left(\sum_{0 \leq i < j \leq n-1} \left| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right|^2 + \sum_{0 \leq i < j \leq n-1} \left| \sqrt{\frac{\alpha_i}{\alpha_j}} y_i - \sqrt{\frac{\alpha_j}{\alpha_i}} y_j \right|^2 \right) + \left| \sum_{i=0}^n (x_i - y_i) \right|^2 \right) \\ &\quad (\text{by (4.1)}) \\ &\geq \sum_{i,j=0}^{n-1} \tau \left(\psi \left(\left| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right|^2 \right) \right) + \sum_{i,j=0}^{n-1} \tau \left(\psi \left(\left| \sqrt{\frac{\alpha_i}{\alpha_j}} y_i - \sqrt{\frac{\alpha_j}{\alpha_i}} y_j \right|^2 \right) \right) \\ &+ \tau \left(\psi \left(\left| \sum_{i=0}^n (x_i - y_i) \right|^2 \right) \right) \quad (\text{by (2.3)}) \\ &= \sum_{i,j=0}^{n-1} \tau \left(\varphi \left(\left| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right| \right) \right) + \sum_{i,j=0}^{n-1} \tau \left(\varphi \left(\left| \sqrt{\frac{\alpha_i}{\alpha_j}} y_i - \sqrt{\frac{\alpha_j}{\alpha_i}} y_j \right| \right) \right) \\ &+ \tau \left(\varphi \left(\left| \sum_{i=0}^n (x_i - y_i) \right| \right) \right) \end{aligned}$$

□

Corollary 4.2. *Let x_0, \dots, x_{n-1} be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \frac{1}{\alpha_j} = 1$. Let φ be a continuous increasing function on \mathbb{R}_+ with $\varphi(0) = 0$ such that $\psi(t) = \varphi(\sqrt{t})$ is convex. Then*

$$\sum_{j=0}^{n-1} \frac{1}{\alpha_j} \tau(\varphi(|\alpha_i x_i|)) \geq \sum_{0 \leq i < j \leq n-1} \tau \left(\varphi \left(\left| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right| \right) \right) + \tau \left(\varphi \left(\left\| \sum_{i=0}^{n-1} x_i \right\| \right) \right). \quad (4.4)$$

If $\psi(t)$ is concave, then the reverse of inequality 4.4 holds.

Proof. It is sufficient that put $y_1 = \dots = y_n = 0$ in Theorem 4.1. \square

Applying Corollary 4.2 for the function $\varphi(t) = t^p$ ($2 \leq p < \infty$) and $\varphi(t) = t^p$ ($0 < p \leq 2$) respectively, we get the following corollary.

Corollary 4.3. *Let x_0, \dots, x_{n-1} be τ -measurable operators and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \frac{1}{\alpha_j} = 1$. Then*

$$\sum_{i=0}^{n-1} \alpha_i^{p-1} \|x_i\|^p \geq \sum_{0 \leq i < j \leq n-1} \left\| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right\|_p^p + \left\| \sum_{i=0}^{n-1} x_i \right\|_p^p$$

for $2 \leq p < \infty$, and

$$\sum_{0 \leq i < j \leq n-1} \left\| \sqrt{\frac{\alpha_i}{\alpha_j}} x_i - \sqrt{\frac{\alpha_j}{\alpha_i}} x_j \right\|_p^p + \left\| \sum_{i=0}^{n-1} x_i \right\|_p^p \geq \sum_{i=0}^{n-1} \alpha_i^{p-1} \|x_i\|^p$$

for $0 < p \leq 2$.

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